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# Double boundary layers on an electrically conducting plane surface in a rotating environment

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A class of nonlinear boundary layers in a hydromagnetic flow under differential rotation is analysed. The function of these layers is to provide smooth transition from the conditions within the electrically conducting container to those in the region far away from the boundary through the flow regime. The structure of the double-decker boundary layer depends on the diffusivity of the fluid, the conductance of the rigid boundary and the relative strength of the applied magnetic field. The method of multiple scales is used to obtain uniformly valid solutions with the conductance of the container varying from zero to infinity. It is found that even a small differential rotation ($\epsilon \rightarrow 0$) can induce perturbations of order $\epsilon^{\frac{1}{2}}$ or of order unity in the field functions.

## 1. Introduction

During the past few years there has developed a great deal of interest in the study of hydromagnetics of rotating fluids. This interest is motivated partly by its possible application to situations of geophysical and astrophysical interest and partly by the desire to gain understanding of the fluid behaviour for various conditions and configurations.

It is well known that the Ekman boundary layer can induce an axial flow of fluid, called the Ekman suction or Ekman pumping. This in turn drives a meridional circulation. The influence of an axial magnetic field on the Ekman layer was studied originally by Gilman & Benton (1968) and more recently by Chawla (1976). It was revealed by these analyses that the hydromagnetic forces act in concert with the Ekman pumping to control the interior of the rotating fluid. These hydromagnetic forces are generated by a meridional circulation of electric current between different regions of fluid flow. This circulation of current is mainly responsible for the establishment of the magnetic diffusion region outside the Ekman–Hartmann layer.

In the case in which the fluid rotates in contact with an electrically conducting container, the electric current leaks out of the fluid into the conducting boundary. For a sufficiently high conductance of the container surface, the evolution of the double-decker hydromagnetic boundary layer is brought about not by viscous stresses but by electromagnetic coupling. The overall effect of the relative strength of the applied magnetic field on hydromagnetic flow in contact with a differentially rotating insulating boundary was considered by Chawla (1976). The specific purpose of the present paper is to investigate how the electrical conductivity of the container controls the structure of the nonlinear hydromagnetic boundary layers. A linear analysis was given by Loper (1970). This, however, precludes the discussion of the outer magnetic diffusion region. But the dynamics of the outer region are crucial for determining the true character of the hydromagnetic coupling. In this context, the strength of the coupling between the interior flow and the boundary is determined not by the ratio of the boundary conductivity to that of the fluid, but rather by the ratio of the boundary conductance to that of the fluid in one Ekman (or magnetic Ekman) depth. The intensity of hydromagnetic coupling afforded by the conductance of the container is measured by the azimuthal (toroidal) magnetic field perturbation at the interface.

Bullard (1950) has suggested that for the interior of the earth electromagnetic coupling is probably more important than the viscous stresses. An additional motivation for the present analysis stems from this suggestion. The electromagnetic coupling gives rise to the possibility of attributing the irregular changes in the length of a day to the rotational momentum transfer between the core and the mantle of the earth. The dynamical response of the mantle to the changing internal magnetic field of the earth is determined by the core–mantle boundary value of this field. The boundary values of the field are affected by the distribution of electrical conductivity in the overlying mantle.

## 2. The mathematical formulation

We consider the situation in which a homogeneous fluid of constant viscosity $\nu$ and magnetic diffusivity $\eta$ fills the half-space $Z > 0$, while the lower half-space $Z \leqslant 0$ is occupied by a rigid electrically conducting container. The container is rotating with constant angular velocity $\Omega(1+\epsilon)$ $(0 < \epsilon \ll 1)$, whereas the fluid at infinity is in a state of rigid-body rotation of angular velocity $\Omega$ in the same sense. A uniform magnetic field of strength $H_0$ is applied in the far field and is constrained to be in the axial direction at large distances from the interface. Following Loper (1970), we assume the electrical conductivity of the container to be a non-negative function of distance from the interface $Z = 0$ subject to the condition that, if the conductivity is zero at a finite distance from $Z = 0$, it must be so at all greater distances. We shall, however, relax the second condition imposed by Loper (1970), which requires the conductance of the container to be finite. We propose to derive uniformly valid solutions of the physical problem for all values of the conductance from zero (insulating boundary) to infinity (perfectly conducting boundary).

The steady equations governing the fluid velocity $\mathbf{V}$, the magnetic field $\mathbf{H}$ and the hydromagnetic pressure $p$, written in an inertial frame, are

$$\nabla . (\tfrac{1}{2}\mathbf{V} . \mathbf{V}) + (\nabla \times \mathbf{V}) \times \mathbf{V} = -\rho^{-1}\nabla p - \nu\nabla \times (\nabla \times \mathbf{V}) + (\mu/\rho)(\nabla \times \mathbf{H}) \times \mathbf{H}, \tag{2.1}$$

$$\nabla \times (\mathbf{V} \times \mathbf{H}) = \eta \nabla \times (\nabla \times \mathbf{H}), \tag{2.2}$$

$$\nabla . \mathbf{V} = 0, \quad \nabla . \mathbf{H} = 0, \tag{2.3}$$

where $\rho$ is the density and $\mu$ is the magnetic permeability of the fluid. The equations governing the steady magnetic field within the solid are

$$\mu\sigma_p \nabla \times (\mathbf{V}_p \times \mathbf{H}_p) + \sigma_p^{-1} \nabla\sigma_p \times (\nabla \times \mathbf{H}_p) = \nabla \times (\nabla \times \mathbf{H}_p), \tag{2.4}$$

$$\nabla . \mathbf{H}_p = 0, \tag{2.5}$$

where $\sigma_p(z)$ is the electrical conductivity at any point and $\mathbf{V}_p$ is the azimuthal velocity of the rotating solid. Moreover, we take the magnetic permeabilities of the solid and the fluid to be the same.

In cylindrical polar co-ordinates $(r, \theta, Z)$ the appropriate boundary conditions are

$$\mathbf{V}.\hat{\mathbf{r}} \to 0, \quad \mathbf{V}.\hat{\boldsymbol{\theta}} \to r\Omega, \quad \mathbf{H} \to H_0 \hat{\mathbf{z}} \quad \text{as} \quad Z \to \infty, \tag{2.6a}$$

$$\mathbf{V} = r\Omega(1+\epsilon)\hat{\boldsymbol{\theta}}, \quad \mathbf{H} = \mathbf{H}_p, \quad (\sigma_p \nabla \times \mathbf{H}).(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}) = (\sigma_f \nabla \times \mathbf{H}_p).(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}) \quad \text{on} \quad Z = 0, \tag{2.6b}$$

$$\mathbf{H}_p.\hat{\mathbf{r}} \to 0, \quad \mathbf{H}_p.\hat{\boldsymbol{\theta}} \to 0 \quad \text{as} \quad Z \to -\infty, \tag{2.6c}$$

where $\sigma_f$ is the electrical conductivity of the fluid and $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\mathbf{z}}$ are unit vectors in the $r$, $\theta$ and $Z$ directions respectively. The conditions (2.6*b*) are necessary for the continuity of the magnetic field and the tangential components of the electric field across the fluid–solid interface. The conditions (2.6*a*, *c*), on the other hand, provide the proper balance in the far field.

For consistency with the axial symmetry and the continuity equations, we define

$$\mathbf{V} = r\Omega[F_z \hat{\mathbf{r}} + (1+G)\hat{\boldsymbol{\theta}}] - 2(\nu\Omega)^{\frac{1}{2}} F \hat{\mathbf{z}}, \tag{2.7}$$

$$\mathbf{H} = -\frac{H_0 r}{\eta}(\nu\Omega)^{\frac{1}{2}}[N_z \hat{\mathbf{r}} + M\hat{\boldsymbol{\theta}}] + H_0\left(1 + \frac{2\nu}{\eta} N\right)\hat{\mathbf{z}}, \tag{2.8}$$

$$p = \tfrac{1}{2} r^2 \Omega^2 + P, \tag{2.9}$$

$$\mathbf{H}_p = -\frac{H_0 r}{\eta}(\nu\Omega)^{\frac{1}{2}}\left[\frac{1}{L}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \overline{N}_\xi \hat{\mathbf{r}} + \overline{M}\hat{\boldsymbol{\theta}}\right] + H_0\left(1 + \frac{2\nu}{\eta}\overline{N}\right)\hat{\mathbf{z}}, \tag{2.10}$$

where $F$, $G$, $M$, $N$ and $P$ are functions of the dimensionless variable $z = (\Omega/\nu)^{\frac{1}{2}} Z$, and $\overline{M}$ and $\overline{N}$ are functions of the dimensionless variable $\xi = Z/L$. Here $L$ is the thickness of a slab of constant electrical conductivity $\sigma_p(0)$ whose conductance equals that of the actual conducting boundary, i.e.

$$L\sigma_p(0) = \int_{-\infty}^{0} \sigma_p(Z)\, dZ. \tag{2.11}$$

We substitute (2.7)–(2.10) in (2.1), (2.2) and (2.4) and get

$$F_{zzz} + 2G - 2\lambda N_{zz} = F_z^2 - 2FF_{zz} - G^2 - 2\lambda\sigma(N_z^2 - 2NN_{zz} - M^2), \tag{2.12}$$

$$G_{zz} - 2F_z - 2\lambda M_z = 2(GF_z - FG_z) - 4\lambda\sigma(MN_z - NM_z), \tag{2.13}$$

$$M_{zz} - G_z = 2\sigma(NG_z - FM_z), \tag{2.14}$$

$$N_{zz} - F_z = 2\sigma(NF_z - FN_z), \tag{2.15}$$

$$\sigma_p \overline{M}_{\xi\xi} = \sigma_{p\xi} \overline{M}_\xi, \quad \overline{N}_{\xi\xi} = 0, \tag{2.16}$$

where $\lambda$ and $\sigma$ are dimensionless parameters defined by

$$\lambda = \mu H_0^2 / 2\rho\eta\Omega, \quad \sigma = \nu/\eta. \tag{2.17}$$

Here $\sigma$ is the magnetic Prandtl number and $\lambda$ measures the strength of the magnetic force relative to the centrifugal force. In terms of $M$ and $N$, the current density vector is

$$\mathbf{J} = (H_0/\eta)[r\Omega M_z \hat{\mathbf{r}} - r\Omega N_{zz} \hat{\boldsymbol{\theta}} - 2(\nu\Omega)^{\frac{1}{2}} M \hat{\mathbf{z}}]. \tag{2.18}$$

The magnetic field within the body of the container satisfies (2.16) and is given by

$$\bar{M}(\xi) = M(0)\left[1 - \frac{1}{\sigma_p(0)}\int_\xi^0 \sigma_p(\xi)\,d\xi\right], \quad \bar{N}(\xi) = 0. \tag{2.19}$$

Consistent with (2.7), (2.8) and (2.19), the boundary conditions (2.6) on the velocity and magnetic fields within the fluid are transformed into

$$F_z(0) = 0, \quad F(0) = 0, \quad G(0) = \epsilon, \quad F(\infty) = 0 = G(\infty), \tag{2.20a}$$

$$N_z(0) = 0, \quad \phi M_z(0) = M(0), \quad M(\infty) = 0 = N(\infty), \tag{2.20b}$$

where

$$\phi = \left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}} \frac{L\sigma_p(0)}{\sigma_f}. \tag{2.21}$$

For an insulating boundary $\phi = 0$, whereas for a perfectly conducting container $\phi = \infty$. The analysis of Loper (1970) implies that $M(\infty), N_z(\infty) \neq 0$. This leaves unbalanced terms in (2.12), (2.13) and (2.15).

In this paper we are concerned with very small differential rotation ($\epsilon \to 0$). If we attempt a solution of the set of equations (2.12)–(2.15) by writing down Taylor-series expansions for $F$, $G$, $M$ and $N$ in powers of $\epsilon$, powers of $z$ appear in the analysis of higher perturbations. Thus, while such a solution satisfies all the boundary conditions on $z = 0$, it fails to satisfy those in the far field. In order to avoid such difficulties of the singular expansion, we introduce two length scales at the outset. Such a procedure is effective, though complicated, primarily because the hydromagnetic flow under consideration has a distinct double-layered structure. The introduction of the additional length scale also provides a complete interaction between the two regions of fluid flow. In the nonlinear treatment, the scaling and the exact form of different hydromagnetic layers depend critically on the order of magnitude of $\phi$.

## 3. Solution when $\phi = O(1)$

It follows from the analysis of Chawla (1976) that, for an insulating container, the outer layer is $O(\epsilon^{-1})$ times as thick as the inner boundary layer. It is therefore appropriate to introduce $\zeta = \epsilon z$ as the additional independent variable. At each stage, the $\zeta$ dependence of the field functions is chosen to suppress any singularities which appear in the perturbation process.

It is shown in Chawla (1976) that, for $\phi = 0$, the nonlinear changes in the applied axial magnetic field communicated to the container surface through the boundary layers are of order unity. Anticipating the other field functions to be of order $\epsilon$, we set

$$F = \epsilon f, \quad G = \epsilon g, \quad M = \epsilon m, \quad N = c + \epsilon n. \tag{3.1}$$

If we now substitute (3.1) in (2.15) and seek a solution of the form

$$f(z, \zeta, \epsilon) = \sum_{n=0}^{\infty} \epsilon^n f^n(z, \zeta), \quad \text{etc.,} \tag{3.2}$$

we immediately find that $c$ is an arbitrary function of $\zeta$. The problem now is to solve for $f(z, \zeta)$, $g(z, \zeta)$, $m(z, \zeta)$ and $n(z, \zeta)$, where

$$f_{zzz} + 2g - 2\lambda(1+2\sigma c)\,n_{zz} = \epsilon[f_z^2 - 2ff_{zz} - g^2 - 3f_{zz\zeta} + 4\lambda(1+2\sigma c)\,n_{z\zeta} + 2\lambda(1+2\sigma c)\,c_{\zeta\zeta} - 2\lambda\sigma(n_z^2 + 2n_z c_\zeta + c_\zeta^2 - 2nn_{zz} - m^2)]$$

$$+\epsilon^2[2f_z f_\zeta - 4ff_{z\zeta} - 3f_{z\zeta\zeta} + 2\lambda(1+2\sigma c)\,n_{\zeta\zeta} - 4\lambda\sigma(n_z n_\zeta - 2nn_{z\zeta} + c_\zeta n_\zeta)]$$
$$+\epsilon^3[f_\zeta^2 - 2ff_{\zeta\zeta} - f_{\zeta\zeta\zeta} - 2\lambda\sigma(n_\zeta^2 - 2nn_{\zeta\zeta})], \tag{3.3}$$

$$g_{zz} - 2f_z - 2\lambda(1+2\sigma c)\,m_z = \epsilon[2f_\zeta + 2(gf_z - fg_z) + 2\lambda(1+2\sigma c)\,m_\zeta - 4\lambda\sigma(mn_z + mc_\zeta - nm_z)] + \epsilon^2[2(gf_\zeta - fg_\zeta) - 4\lambda\sigma(mn_\zeta - nm_\zeta)], \tag{3.4}$$

$$m_{zz} - (1+2\sigma c)\,g_z = \epsilon[(1+2\sigma c)\,g_\zeta - 2m_{z\zeta} + 2\sigma(ng_z - fm_z)] + \epsilon^2[2\sigma(ng_\zeta - fm_\zeta) - m_{\zeta\zeta}], \tag{3.5}$$

$$n_{zz} - (1+2\sigma c)f_z = \epsilon[(1+2\sigma c)f_\zeta - 2n_{z\zeta} - c_{\zeta\zeta} + 2\sigma(nf_z - fn_z - fc_\zeta)] + \epsilon^2[2\sigma(nf_\zeta - fn_\zeta) - n_{\zeta\zeta}], \tag{3.6}$$

with

$$\left.\begin{aligned} &f(0,0) = 0, \quad f_z(0,0) + \epsilon f_\zeta(0,0) = 0, \quad g(0,0) = 1, \\ &n_z(0,0) + c_\zeta(0,0) + \epsilon n_\zeta(0,0) = 0, \\ &\phi[m_z(0,0) + \epsilon m_\zeta(0,0)] = m(0,0), \end{aligned}\right\} \tag{3.7}$$

$$\left.\begin{aligned} &f_z(\infty,\infty) + \epsilon f_\zeta(\infty,\infty) = 0, \quad g(\infty,\infty) = 0, \\ &m(\infty,\infty) = 0, \quad c(\infty) + \epsilon n(\infty,\infty) = 0. \end{aligned}\right\} \tag{3.8}$$

Substituting (3.2) in (3.3)–(3.6) and equating the coefficients of like powers of $\epsilon$ on both sides, we have

$$f^0_{zzz} + 2g^0 - 2\lambda(1+2\sigma c)\,n^0_{zz} = 0, \tag{3.9a}$$

$$g^0_{zz} - 2f^0_z - 2\lambda(1+2\sigma c)\,m^0_z = 0, \tag{3.9b}$$

$$m^0_{zz} - (1+2\sigma c)\,g^0_z = 0, \tag{3.9c}$$

$$n^0_{zz} - (1+2\sigma c)f^0_z = 0, \tag{3.9d}$$

$$f^1_{zzz} + 2g^1 - 2\lambda(1+2\sigma c)\,n^1_{zz} = f^0_z - 2f^0 f^0_{zz} - g^{02} - 2\lambda\sigma[n^{02}_z - 2n^0 n^0_{zz} + 2n^0_z c_\zeta + c_\zeta^2 - m^{02}] - 3f^0_{zz\zeta} + 4\lambda(1+2\sigma c)\,n^0_{z\zeta} + 2\lambda(1+2\sigma c)\,c_{\zeta\zeta}, \tag{3.10a}$$

$$g^1_{zz} - 2f^1_z - 2\lambda(1+2\sigma c)\,m^1_z = 2f^0_\zeta + 2(g^0 f^0_z - f^0 g^0_z) - 2g^0_{z\zeta} + 2\lambda(1+2\sigma c)\,m^0_\zeta - 4\lambda\sigma(m^0 n^0_\zeta + m^0 c_\zeta - n^0 m^0_z), \tag{3.10b}$$

$$m^1_{zz} - (1+2\sigma c)\,g^1_z = (1+2\sigma c)\,g^0_\zeta - 2m^0_{\zeta\zeta} + 2\sigma(n^0 g^0_z - f^0 m^0_z), \tag{3.10c}$$

$$n^1_{zz} - (1+2\sigma c)f^1_z = (1+2\sigma c)f^0_\zeta - 2n^0_{z\zeta} - c_{\zeta\zeta} + 2\sigma(n^0 f^0_z - f^0 n^0_z - f^0 c_\zeta), \tag{3.10d}$$

and so on. The solution of the set (3.9) is

$$f^0 = a^0 - \frac{1}{2}\left[\frac{A^0 \exp(-sz)}{s} + \frac{A^{0*}}{s^*}\exp(-s^*z)\right], \tag{3.11a}$$

$$g^0 = \frac{1}{2i}[A^0 \exp(-sz) - A^{0*}\exp(-s^*z)], \tag{3.11b}$$

$$m^0 = d^0 - \frac{1+2\sigma c}{2i}\left[\frac{A^0}{s}\exp(-sz) - \frac{A^{0*}}{s^*}\exp(-s^*z)\right], \tag{3.11c}$$

$$n^0 = c^0 + \frac{1+2\sigma c}{2}\left[\frac{A^0}{s^2}\exp(-sz) + \frac{A^{0*}}{s^{*2}}\exp(-s^*z)\right], \tag{3.11d}$$

where

$$s = [2i + 2\lambda(1+2\sigma c)^2]^{\frac{1}{2}}, \tag{3.12}$$

and $a^0$, $A^0$, $c^0$ and $d^0$, like $c$, are arbitrary functions of $\zeta$. Also, an asterisk denotes the complex conjugate of the function under it.

In writing the solution (3.11) care was taken that powers of $z$ did not appear. Otherwise these would propagate into the higher approximations with larger values. For the same reason, we prevent terms independent of $z$ from appearing on the right sides of (3.10*b*, *d*). This yields

$$2a^0_\zeta + 2\lambda d^0_\zeta + 4\lambda\sigma(cd^0_\zeta - d^0 c_\zeta) = 0, \tag{3.13}$$

$$a^0_\zeta + 2\sigma(ca^0_\zeta - a^0 c_\zeta) - c_{\zeta\zeta} = 0. \tag{3.14}$$

By eliminating $g^1$, $m^1$ and $n^1$ from the set (3.10), we avoid terms of the form

$$z^n \exp(-sz) \quad (n \geqslant 0),$$

so that we must have

$$s^2 A^0_\zeta + [a^0 m^0 + \lambda\sigma(1+2\sigma c)(4cm^0 + 3c_\zeta + 2id^0 + 2a^0(1+2\sigma c))]A^0 = 0, \tag{3.15$a$}$$

with

$$A^0(0) = i. \tag{3.15$b$}$$

Moreover, it is evident from (3.10*a*) that the function $g^1(z,\zeta)$ must be of the form

$$g^1(z,\zeta) = b^1(\zeta) + g^1_1(z,\zeta), \tag{3.16}$$

where

$$2b^1 = 2\lambda(1+2\sigma c)c_{\zeta\zeta} - 2\lambda\sigma(c^2_\zeta - d^{02}). \tag{3.17}$$

For uniform validity of the second approximation, it now follows from (3.5) that

$$b^1_\zeta + 2\sigma(cb^1_\zeta - a^0 d^0_\zeta) - d^0_{\zeta\zeta} = 0. \tag{3.18}$$

The solution of the system of nonlinear differential equations (3.13), (3.14), (3.17) and (3.18), with appropriate boundary conditions, provides details about the physical structure of the outer layer whereas the solution of the set (3.15) gives minor modifications in the thickness of the inner boundary layer. The solution (3.11) gives a unified representation (to order $\epsilon$) of the flow and magnetic field functions in the whole of the flow region. Clearly the inner boundary layer (called the Ekman–Hartmann layer) results from the viscous–centrifugal–magnetic force balance near the rigid boundary. The thickness of this layer decreases with increasing $\lambda$. The conducting boundary supports an azimuthal component of the magnetic field through the thickness of the Ekman–Hartmann layer.

The dynamics of the outer region are crucial for determining the intensity of the hydromagnetic coupling between different regions of fluid flow and the interior of the container. Writing $p = a^{0\prime} + ib^1$ and $q = c' + id^0$, where a prime denotes differentiation with respect to $\zeta$, (3.13), (3.14), (3.17) and (3.18) are combined as

$$2ip + 2\lambda q' = 2\sigma\lambda(q^2 - 2cq'), \tag{3.19$a$}$$

$$q'' - p' = 2\sigma(cp' - a^0 q'). \tag{3.19$b$}$$

Also the boundary conditions (3.7) and (3.8) yield

$$a^0(0) = \frac{i}{2}\left[\frac{1}{s(0)} - \frac{1}{s^*(0)}\right], \quad q(0) = ik\phi + \frac{ik}{s(0)}, \tag{3.20$a$}$$

$$p(\infty) = 0, \quad q(\infty) = 0, \quad c(\infty) = 0, \tag{3.20$b$}$$

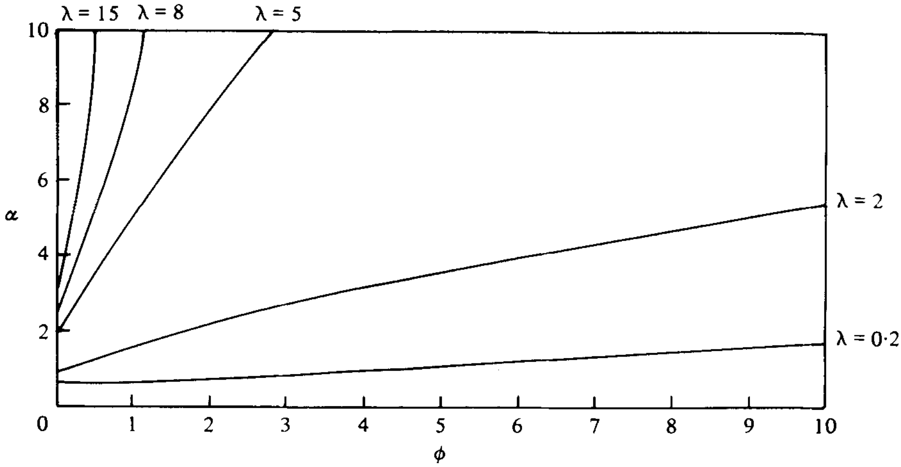


FIGURE 1. Variation of $\alpha = a^0(\infty)$ with $\phi$ for $\lambda = 0{\cdot}2,\ 2,\ 5,\ 8,\ 15$.

where

$$k = 1 + 2\sigma c(0), \quad s(0) = [2i + 2\lambda k^2]^{\frac{1}{2}}. \tag{3.21}$$

It is evident from (3.19) that the induced electromagnetic body force dominates the nonlinear part of the inertia in the outer region. In addition to providing the necessary balance between magnetic diffusion and magnetic convection, the outer region (called the magnetic diffusion region) serves to balance the induced electromagnetic body force and the induced centrifugal force.

The set of equations (3.19)–(3.20), in the case $\phi = 0$, has been solved by Chawla (1976) using a method due to Fettis (1955). We omit details and give below only the values of $\alpha$ ($= a^0(\infty)$) and $k$ (based on a four-term solution) for small and large values of the dimensionless parameter $\lambda$ (with $\phi = O(1)$):

$$\alpha = \begin{cases} \dfrac{1}{2}\left[1 + \dfrac{3+8\phi}{8}\lambda + \dfrac{-77-256\phi-256\phi^2+96\phi^3+48\phi^4}{384}\lambda^2 + O(\lambda^3)\right] & (\lambda \text{ small}), \quad (3.22a)\\ \lambda[\phi + (2\lambda)^{-\frac{1}{2}}] + O(\lambda^{-1}) & (\lambda \text{ large}), \quad (3.22b)\end{cases}$$

$$k = \begin{cases} \dfrac{1}{2}\left[1 - \dfrac{1+2\phi}{4}\lambda + \dfrac{59+187\phi+199\phi^2+12\phi^3+6\phi^4}{192}\lambda^2 + O(\lambda^3)\right] & (\lambda \text{ small}), \quad (3.23a)\\ 1 + O(\lambda^{-\frac{1}{2}}) & (\lambda \text{ large}). \quad (3.23b)\end{cases}$$

$\epsilon\alpha$ gives the axial inflow at infinity whereas $k$ gives the normal magnetic field induced within the conducting container. The azimuthal magnetic field supported over the conducting boundary is given by $\epsilon k\phi$. $\alpha$ and $k$ are plotted *vs.* $\phi$ in figures 1 and 2 respectively.

The above analysis clearly establishes the existence of a double-layered structure governing the hydromagnetic flow induced by differential rotation. The function of the two layers is to provide a smooth transition in the applied magnetic field from its value outside the magnetic diffusion region (MDR) to that within the container. Any fluid flow in the MDR adjusts itself to the velocity of the rotating container across the Ekman–Hartmann layer (EHL). The thickness of the EHL is $O(l_r^{-1})$, where

$$l_r = [(1 + \lambda^2 k^4)^{\frac{1}{2}} + \lambda k^2]^{\frac{1}{2}}, \tag{3.24}$$

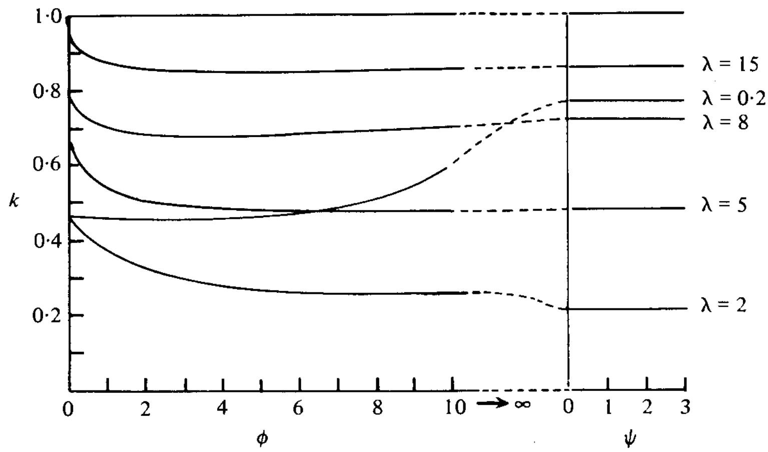


FIGURE 2. Variation of $k$ with $\phi$ and $\psi$ for $\lambda$ = 0·2, 2, 5, 8, 15.

and $k$ varies from $\frac{1}{2}$ to 1 as the strength of the applied magnetic field is increased. On the other hand, in general $k$ decreases as the electrical conductivity of the container is increased. Thus the EHL will be thicker for a conducting boundary than that for a non-conducting one.

The overall effect of the relative magnetic field strength on the structure of the two regions of fluid flow in their nonlinear forms has already been discussed in Chawla (1976). We therefore confine ourselves to considering the effect of the electrical conductivity of the container on the hydromagnetic interaction.

It is evident from figure 1 that the inflow velocity at infinity increases with $\phi$ whereas the thickness of the outer layer (MDR), which is of order $(1+\lambda^2)/2\alpha\sigma\epsilon$, decreases as $\phi$ is increased. It is therefore natural that the increased influx into the outer region is balanced by increased centrifugal action, so that the azimuthal velocity just outside the EHL increases with $\phi$. This fact is clearly brought out in the next section.

In addition to its ability to induce mass flux into the MDR, the EHL also generates electric current. The Hartmann current flowing into the outer region is given by

$$\epsilon m(0) = \begin{cases} \dfrac{\epsilon}{4}\left[1+2\phi-\dfrac{1+4\phi}{8}\lambda+\dfrac{100+151\phi+199\phi^2+12\phi^3+6\phi^4}{192}\lambda^2+O(\lambda^3)\right] & (\lambda \text{ small}), \quad (3.25a) \\ \epsilon[\phi+(2\lambda)^{-\frac{1}{2}}+O(\lambda^{-\frac{3}{2}})] \quad (\lambda \text{ large}). & (3.25b) \end{cases}$$

The axial current leaking into the conducting boundary, which is responsible for the toroidal magnetic field at the interface, is $\epsilon k\phi$. A comparison between (3.23) and (3.25) shows that the EHL can support a stronger toroidal magnetic field than can the conducting boundary. In the MDR, the toroidal field stretches out into the axial field of strength $H_0$, whereas within the container it stretches out, in the opposite sense, into an axial field of strength $kH_0$.

The diffusive effects of the finite resistivity of the fluid tend to spread the electric current pattern axially within the MDR and thicken the boundary layer. As the conductance of the boundary is increased, more and more current leaks into the boundary, 'pulling' the edge of the outer layer with it. For a sufficiently large value of

$\phi$ the spreading and pulling achieve a balance and give rise to a different type of outer boundary layer. This is discussed in the next section.

## 4. Solution when $\phi = O(\epsilon^{-\frac{1}{2}})$

We note from (3.24*a*) that, for $\phi = O(\epsilon^{-\frac{1}{2}})$, $\alpha = O(\epsilon^{\frac{1}{2}})$, so that the thickness of the outer layer is of order $\epsilon^{-\frac{1}{2}}$. The appropriate additional length scale in this case would be $x = \epsilon^{\frac{1}{2}}z$. Writing

$$\left.\begin{aligned} \phi_1 = \epsilon^{\frac{1}{2}}\phi, \quad F = \epsilon^{\frac{1}{2}}f(z,x), \quad G = \epsilon g(z,x), \\ M = \epsilon^{\frac{1}{2}}m(z,x), \quad N = c(x)+\epsilon^{\frac{1}{2}}n(z,x), \end{aligned}\right\} \tag{4.1}$$

we assume a solution of the form

$$f = \sum_{n=0}^{\infty} \epsilon^{\frac{1}{2}n} f^n(x,z) \quad \text{etc.}$$

Proceeding as in the last section, we immediately find that

$$f^0 = a^0(x), \quad m^0 = d^0(x), \quad n^0 = c^0(x), \tag{4.2}$$

$$f^1 = a^1(x) - \frac{1}{2}\left[\frac{A(x)}{s}\exp(-sz) + \frac{A^*(x)}{s^*}\exp(-s^*z)\right], \tag{4.3a}$$

$$g^0 = b^0(x) + \frac{1}{2i}[A\exp(-sz) - A^*\exp(-s^*z)], \tag{4.3b}$$

$$m^1 = d^1(x) - \frac{1+2\sigma c}{2i}\left[\frac{A}{s}\exp(-sz) - \frac{A^*}{s^*}\exp(-s^*z)\right], \tag{4.3c}$$

$$n^1 = c^1(x) + \frac{1+2\sigma c}{2}\left[\frac{A}{s^2}\exp(-sz) + \frac{A^*}{s^{*2}}\exp(-s^*z)\right], \tag{4.3d}$$

with

$$s = [2i + 2\lambda(1+2\sigma c)^2]^{\frac{1}{2}},$$

where $a^0$, $b^0$, $c^0$, $d^0$, $A$, $a^1$, $b^1$, $d^1$ and $c^1$ are arbitrary functions of $x$. For a uniformly valid solution, the functions $a^0$, $b^0$, $c^0$ and $d^0$ are given by

$$2a^0_x + 2\lambda d^0_x - 4\lambda\sigma(d^0 c_x - c d^0_x) = 0, \tag{4.4}$$

$$a^0_x - c_{xx} + 2\sigma(c a^0_x - a^0 c_x) = 0, \tag{4.5}$$

$$2b^0 = 2\lambda c_{xx} - 2\lambda\sigma(c^2_x - 2c c_{xx} - d^{02}), \tag{4.6}$$

$$(1+2\sigma c)\, d^0_x - d^0_{xx} - 2\sigma a^0 d^0_x = 0. \tag{4.7}$$

Expressions (4.1) with (4.2) and (4.3) provide a uniform representation (to order $\epsilon$) of the field functions for the entire region of hydromagnetic interaction. The physical characteristics of the inner EHL do not differ much from those already considered in the previous section. In order to derive the main features of the dynamics of the outer layer we set

$$a^0\sigma^{\frac{1}{2}} = a, \quad \sigma c = C, \quad d^0\sigma^{\frac{1}{2}} = d, \quad x\sigma^{\frac{1}{2}} = y, \tag{4.8}$$

$$\phi_1\sigma^{\frac{1}{2}} = \psi, \quad p = a_y + ib^0, \quad q = C_y + id \tag{4.9}$$

in (4.4)–(4.7) and get

$$2ip + 2\lambda q_y = 2\lambda(q^2 - 2Cq_y), \tag{4.10a}$$

$$q_{yy} - p_y = 2(Cp_y - aq_y), \tag{4.10b}$$

with

$$a(0) = 0,$$

$$2q(0) = \psi[2ik + q_v(0) - q_v^*(0) - kp(0) + kp^*(0)], \tag{4.11a}$$

$$p(\infty) = 0, \quad q(\infty) = 0, \quad C(\infty) = 0. \tag{4.11b}$$

Again we use the Fettis method to solve the above set of equations. A four-term series solution yields the following values of $\alpha$ $(= a(\infty))$, $k$ $(= 1 + 2C(0))$ and $B$ $(= d(0))$ for small and large values of $\lambda$ (with $\psi = O(1)$):

$$\alpha = \begin{cases} (\frac{1}{8}\lambda_1)^{\frac{1}{2}}\,\psi[1 - (\frac{3}{4} + 2\psi_1^2)\,\lambda_1^{\frac{1}{2}} + (\frac{31}{32} + 4\psi_1^2 + 8\psi_1^4)\,\lambda_1 + O(\lambda^{\frac{3}{2}})] & (\lambda \text{ small}), \quad (4.12a) \\ \frac{1}{4}\lambda^{\frac{1}{2}}[1 - (8\psi_1\,\lambda_1^{\frac{1}{2}})^{-1} + O(\lambda^{-1})] & (\lambda \text{ large}), \quad (4.12b) \end{cases}$$

$$k = \begin{cases} 1 - \lambda_1^{\frac{1}{2}} + \frac{5}{4}\lambda_1 + O(\lambda^{\frac{3}{2}}) & (\lambda \text{ small}), \quad (4.13a) \\ 1 + O(\lambda^{-1}) & (\lambda \text{ large}), \quad (4.13b) \end{cases}$$

$$B = \begin{cases} \psi[1 - (1 + 2\psi_1^2)\,\lambda_1^{\frac{1}{2}} + (\frac{5}{4} + \frac{9}{2}\psi_1^2 + 8\psi_1^4)\,\lambda_1 + O(\lambda^{\frac{3}{2}})] & (\lambda \text{ small}), \quad (4.14a) \\ (4\lambda^{\frac{1}{2}})^{-1}\,[1 - (8\psi_1\,\lambda_1^{\frac{1}{2}})^{-1} + O(\lambda^{-1})] & (\lambda \text{ large}), \quad (4.14b) \end{cases}$$

where

$$8^{\frac{1}{2}}\,\lambda = \lambda_1, \quad \psi^2 = 8^{\frac{1}{2}}\psi_1^2. \tag{4.15}$$

The solutions of the differential sets (3.19)–(3.20) and (4.10)–(4.11) between them cover the whole range of values of $\phi$ from zero to infinity. The smooth transition from one solution to the other is evident from figure 2. We note that $\psi$ $(= (\Omega/\eta)^{\frac{1}{2}}\,\sigma_p(0)\,L/\sigma_f)$ is the ratio of the boundary conductance to the fluid conductance of one magnetic Ekman depth. For $\phi = 0$ (insulating boundary), the expressions (3.22) and (3.25) for $\alpha$ and $k$ respectively reduce to the values obtained by Chawla (1976). For $\phi = \infty$ (perfectly conducting container), we have

$$\alpha = \begin{cases} \frac{1}{2}(\frac{1}{2}\lambda_1)^{\frac{1}{4}}\,[1 - \frac{1}{8}\lambda_1^{\frac{1}{2}} + \frac{341}{1152}\lambda_1 + O(\lambda^{\frac{3}{2}})] & (\lambda \text{ small}), \quad (4.16a) \\ \frac{1}{4}\lambda^{\frac{1}{2}} + O(\lambda^{-\frac{3}{2}}) & (\lambda \text{ large}), \quad (4.16b) \end{cases}$$

$$B = \begin{cases} (2/\lambda_1)^{\frac{1}{4}}\,[1 - \frac{3}{8}\lambda_1^{\frac{1}{2}} + \frac{485}{1152}\lambda_1 + O(\lambda^{\frac{3}{2}})] & (\lambda \text{ small}), \quad (4.17a) \\ (4\lambda^{\frac{1}{2}})^{-1} + O(\lambda^{-\frac{5}{2}}) & (\lambda \text{ large}). \quad (4.17b) \end{cases}$$

The value of $k$ remains the same as that given by (4.13).

In the limit $\phi \to \infty$, the bounding surface acts as a perfect conductor and the magnetic field within the rigid container is 'frozen in'. The semi-infinite extent of the fluid forces the magnetic field to rotate with it. The differential rotation steadily twists this magnetic field into the azimuthal direction, resulting in a toroidal field. Since the conductance of the fluid is finite, the differential rotation does not last beyond the double-decker structure. This acts to limit the flow of electric current, giving it a finite value. Loper (1970) assumes the outer region (MDR) to extend spatially to infinity, so that in his case the twisting continues without bound. Thus the approximate calculation by Loper predicts no finite limiting values for the toroidal field in the limit $\phi \to \infty$.

$k$ measures the relative distortions in the applied magnetic field due to the stretching of the vortex lines across the two boundary layers. Compared with the basic applied magnetic field $H_0$ in the far field, the normal field induced within the conducting container is $H_0 k$. We note from (3.26), (4.13) and figure 2 that substantial changes (of order unity) in the basic axial field over the entire width of the flow regime

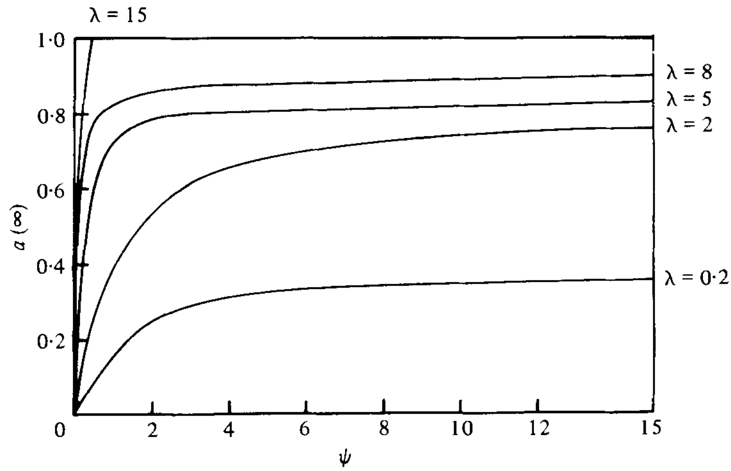


FIGURE 3. Variation of $\alpha = a(\infty)$ with $\psi$ for $\lambda = 0{\cdot}2$, 2, 5, 8, 15.

can be brought about and sustained by even a small differential rotation ($\epsilon \to 0$). Though these changes are affected by the shearing of the fluid within the EHL, the bulk of the distortion takes place in the MDR. The relative distortion of the axial magnetic field is more pronounced for a conducting boundary.

We have already observed that $F(\infty)$ is of order $\epsilon$ for $\phi = O(1)$ whereas for

$$\phi = O(\epsilon^{-\frac{1}{2}})$$

it is of order $\epsilon^{\frac{1}{2}}$; the suction in the far field increases with $\phi$ (see figure 3). Thus, for a given $\lambda$, the thickness $O[(1+\lambda^2)/2\sigma F(\infty)]$ of the MDR decreases from order $\epsilon^{-1}$ to order $\epsilon^{-\frac{1}{2}}$ as $\phi$ increases. For sufficiently large $\lambda$, the thickness of the outer region varies from $O[A_0^3/\epsilon\Omega^2(\nu\eta)^{\frac{1}{2}}]$ to $O[A_0^3/\Omega^2\eta\epsilon^{\frac{1}{2}}]$ as $\phi$ takes on values from zero to infinity, where $A_0$ is the Alfvén velocity $(\mu H_0^2/\rho)^{\frac{1}{2}}$. As already mentioned, the MDR primarily results from the outward magnetic diffusion and the inward magnetic convection. It also serves to balance the induced electromagnetic body force and the inertia of the induced rotation. The leakage of electric current into the conducting boundary tends to inhibit the axial growth of the magnetic diffusion on the one hand and increase the electromagnetic body force on the other. Thus the thickness and the physical character of the MDR go on changing with $\lambda$ and $\phi$. For $\phi = O(\epsilon^{-\frac{1}{2}})$, the MDR is independent of the viscosity of the fluid.

It is natural that the increased inflow into the thinning (outer) layer is balanced by increased centrifugal action. For a highly conducting boundary, the tangential flow is no longer confined to the EHL. Additional layers of the conducting fluid, with tangential velocity of order $\epsilon$, come under the influence of differential rotation as $\phi$ increases. The source of the tangential flow (of order $\epsilon$) induced in the outer layer may be traced to the strong axial current (of order $\epsilon^{\frac{1}{2}}$) drawn out of the EHL. In the outer layer (thickness of order $\epsilon^{-\frac{1}{2}}$), this current turns and interacts with the axial field (of order unity) to generate a tangential electromagnetic body force (of order $\epsilon$). This force is sufficient to drive the edge of the EHL. For $\psi = O(1)$, the angular velocity of the bulk of the outer layer is given by $\Omega(1+\epsilon b^0(0))$, where

$$b^0(0) = \begin{cases} \lambda\psi^2[1 - 2(1+2\psi_1^2)\lambda^{\frac{1}{2}} + O(\lambda)] \quad (\lambda \text{ small}), & (4.18a) \\ 1-(4\psi)^{-1}\lambda^{-\frac{1}{2}} + O(\lambda^{-1}) \quad (\lambda \text{ large}). & (4.18b) \end{cases}$$

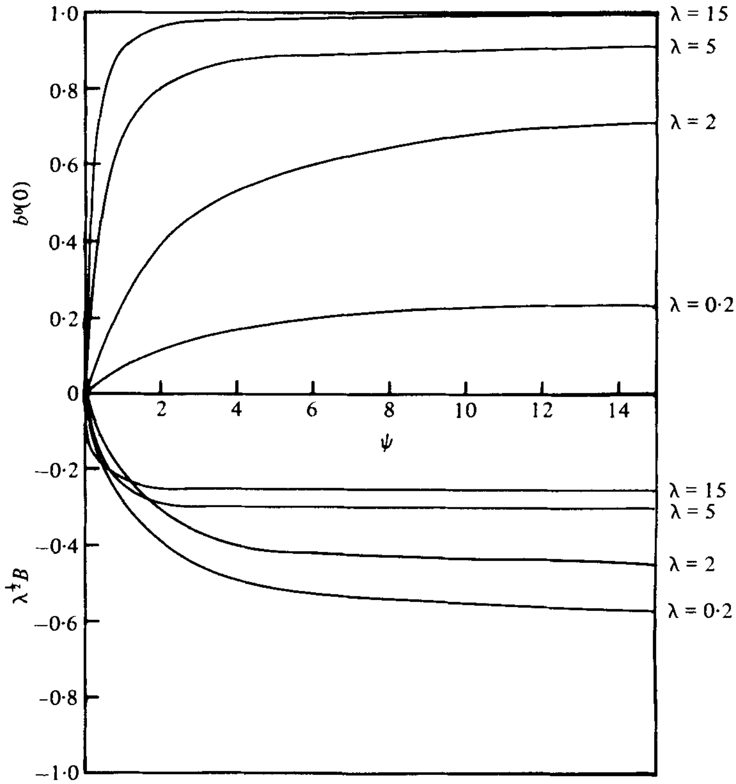


FIGURE 4. Variation of $b^0(0)$ and $(\mu/2\epsilon\rho)^{\frac{1}{2}} H_0/r\Omega$ with $\psi$ for $\lambda = 0{\cdot}2, 2, 5, 15$.

Further, for $\phi = \infty$,

$$b^0(0) = \begin{cases} (2\lambda)^{\frac{1}{2}}[1-\frac{3}{4}\lambda^{\frac{1}{2}} + O(\lambda)] & (\lambda \text{ small}), \quad (4.19a) \\ 1+O(\lambda^{-2}) & (\lambda \text{ large}). \quad (4.19b) \end{cases}$$

The variation of $b^0(0)$ with $\psi$ is shown in figure 4. Evidently this case admits significant velocities within the MDR. In fact the MDR satisfies the various viscous boundary conditions [see (4.11)] and the EHL is no longer necessary for the transition. In the limit $\phi \to \infty$, currents flowing radially inwards within the MDR interact with the applied axial magnetic field to produce a strong body force in the azimuthal direction, and for a sufficiently strong magnetic field the bulk of the fluid within the boundary layers rotates with the container as if in rigid-body rotation at an angular velocity $\Omega(1+\epsilon)$. In view of the fact that the thickness of the outer region decreases as $\phi$ increases, we conclude that the spin-up time is decreased. We infer that, in general, the spin-up is accelerated by a conducting boundary. In contrast, the spin-up is slowed down by increasing the relative strength of the applied magnetic field (see Chawla 1976).

The imposed vertical shear near the more rapidly rotating boundary tilts the axial lines of force. The electrically conducting boundary does not allow the field lines to slip over its surface. This results in the distortions of the basic field penetrating the container. Consequently a stronger axial Hartmann current leaks through the boundary as the conductance of the container is allowed to increase. Associated with the axial current drawn into the MDR and the conducting boundary is a strong azimuthal perturbation component (of order $\epsilon^{\frac{1}{2}}$) of the induced magnetic field. For sufficiently large $\lambda$, the toroidal field permeating the EHL is given by

$$\mu H_\theta = -r\Omega(\tfrac{1}{8}\mu\epsilon\rho)^{\frac{1}{2}}[1-(8\psi)^{-1}\lambda^{-\frac{1}{2}}+O(\lambda^{-1})]. \quad (4.20)$$

The function $\mu^{\frac{1}{2}}H_\theta/(2\rho\epsilon)^{\frac{1}{2}}r\Omega$ is plotted *vs.* $\psi$ in figure 4 for various values of $\lambda$. The evolution (spin-up) of the differentially rotating hydromagnetic flow over an electrically conducting boundary will, therefore, leave a residual azimuthal (toroidal) magnetic field of considerable strength through the thickness of the EHL. The electromagnetic body force associated with this field tends to spin the fluid faster. The viscosity of the fluid plays a negligible role in the spin-up process. The role of viscous stresses in controlling the fluid motion under hydromagnetic interaction is taken over by electromagnetic coupling as the electrical conductivity of the container is increased. This can be ascertained by writing the values of the suction velocity in the far field, the thickness of the boundary layer, the axial current and other physical functions in their dimensional form. In all the functions the magnetic diffusivity of the fluid replaces the kinematic viscosity.

## 5. Concluding remarks

A number of dynamo models have been proposed to explain the solar cycle. All depend on the mechanism first put forward by Parker (1955), whereby the poloidal field is drawn out by differential rotation to give a toroidal field from which a poloidal field with the opposite sense is produced. It is generally assumed that there exists a large toroidal magnetic field within the earth's core associated with the hydromagnetic dynamo. The toroidal field is presumably generated from the main dipole field by differential rotation of the fluid within the core. For the core–mantle interface of the earth $\lambda = O(1)$, $\epsilon = O(10^{-5})$, $\sigma = O(10^{-6})$ and $\phi = O(10^3)$ (see Hide & Roberts 1961). Since $\phi$ is almost of order $\epsilon^{-\frac{1}{2}}$, the results of the last section may be considered to have some relevance to modelling conditions near the poles within the earth's core. But the present analysis concerns a fluid of unbounded extent, both radially and axially. The effect of curvature and non-axisymmetry must be taken into account before qualitative statements can be made about conditions prevailing within the earth's core.

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